Chordal generators and the hydrodynamic normalization for the unit ball

Sebastian Schleißinger*

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Abstract

Let $c \geq 0$ and denote by $\mathcal{K}(\mathbb{H}, c)$ the set of all infinitesimal generators $G : \mathbb{H} \to \mathbb{C}$ on the upper half-plane $\mathbb{H}$ such that $\limsup_{y\to\infty} y \cdot |G(iy)| \leq c$. This class is related to univalent functions $f : \mathbb{H} \to \mathbb{H}$ with hydrodynamic normalization and appears in the so called chordal Loewner equation.

In this paper, we generalize the class $\mathcal{K}(\mathbb{H}, c)$ and the hydrodynamic normalization to the Euclidean unit ball in $\mathbb{C}^n$. The generalization is based on the observation that $G \in \mathcal{K}(\mathbb{H}, c)$ can be characterized by an inequality for the hyperbolic length of $G(z)$.

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Contents

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1 Introduction

1.1 One-parameter semigroups

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the Euclidean unit ball in $\mathbb{C}^n$. In one dimension we write $\mathbb{D} := \mathbb{B}_1$ for the unit disc.

Definition 1.1. A continuous one-real-parameter semigroup of holomorphic functions on $\mathbb{B}_n$ is a map $[0, \infty) \ni t \mapsto \Phi_t \in \mathcal{H}(\mathbb{B}_n, \mathbb{B}_n)$ satisfying the following conditions:

(1) Φ_0 is the identity,

(2) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \geq 0$,

(3) Φ_t tends to the identity locally uniformly in $\mathbb{B}_n$ when t tends to 0.

Given such a semigroup $\{\Phi_t\}_{t\geq 0}$ and a point $z \in \mathbb{B}_n$, then the limit

$$G(z) := \lim_{t\to 0} \frac{\Phi_t(z) - z}{t}$$

exists and the vector field $G : \mathbb{B}_n \to \mathbb{C}^n$, called the *infinitesimal generator*[1] of Φ_t, is a holomorphic function (see, e.g., [Aba92]). We denote by $\mathrm{Inf}(\mathbb{B}_n)$ the set of all infinitesimal generators of semigroups in $\mathbb{B}_n$. For any $z \in \mathbb{B}_n$, the map $w(t) := \Phi_t(z)$ is the solution of the initial value problem

$$\frac{dw(t)}{dt} = G(w(t)), \quad w(0) = z. \tag{1.1}$$

There are various characterizations of holomorphic functions $G : \mathbb{B}_n \to \mathbb{C}^n$ that are infinitesimal generators; see [RS05] (Section 7.3), [BCDM10] (Theorem 0.2), [BES14] (p. 193).

The set $\mathrm{Inf}(\mathbb{D})$, i.e. all infinitesimal generators in the unit disc, can be characterized completely by the Berkson-Porta representation formula (see [BP78]):

$$\mathrm{Inf}(\mathbb{D}) = \{z \mapsto (\tau - z)(1 - \overline{\tau}z)p(z) \,|\, \tau \in \overline{\mathbb{D}},\ p \in \mathcal{H}(\mathbb{D}, \mathbb{C}) \text{ with } \mathrm{Re}(p(z)) \geq 0 \text{ for all } z \in \mathbb{D}\}. \tag{1.2}$$

Remark 1.2. *Let $F : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map. Recall the Denjoy-Wolff theorem (see, e.g., [RS05], Theorem 5.1): If F is not an elliptic automorphism (i.e. an automorphism with exactly one fixed point in $\mathbb{D}$), then there exists one point $\tau \in \overline{\mathbb{D}}$ (the Denjoy-Wolff point of F) such that the iterates F^n converge locally uniformly in $\mathbb{D}$ to the constant map τ.*

If $\{\Phi_t\}_{t\geq 0}$ is a semigroup on $\mathbb{D}$, then we call $\tau \in \overline{\mathbb{D}}$ the Denjoy-Wolff point of $\{\Phi_t\}_{t\geq 0}$ if τ is the Denjoy-Wolff point of Φ_1, which is equivalent to $\lim_{t\to\infty} \Phi_t = \tau$ locally uniformly.
If an infinitesimal generator in the unit disc does not generate a semigroup of elliptic automorphisms of $\mathbb{D}$, then the point $\tau \in \overline{\mathbb{D}}$ from formula (1.2) is exactly the Denjoy-Wolff point of the semigroup.

There are two special cases of infinitesimal generators in $\mathbb{D}$ that have been studied intensively and turned out to be quite useful in Loewner theory and its applications. The two different cases arise from certain normalizations of the Berkson-Porta data τ and p from formula (1.2). In the *radial* case, one considers those elements $G \in \mathrm{Inf}(\mathbb{D})$ whose Berkson-Porta data τ and p satisfy

$$\tau = 0 \qquad \text{and} \qquad p(0) = 1,$$

i.e. $G(z) = -zp(z)$.

This class plays a central role in studying the class S of all univalent functions $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$, $f'(0) = 1$, by the powerful tools of Loewner's theory; see, e.g., [Pom75], Chapter 6. The class of radial generators as well as the class S have been generalized in this context to the polydisc $\mathbb{D}^n$, see [Por87a, Por87b], and to the unit ball $\mathbb{B}_n$, see [GK03] for a collection of several results and references.

The second class, the set of all *chordal* generators[2] , consists of all $G \in \mathrm{Inf}(\mathbb{D})$ whose Berkson-Porta data τ and p satisfy

$$\tau = 1 \qquad \text{and} \qquad \angle \lim_{z\to 1} \frac{p(z)}{z-1} \quad \text{is finite.}$$

The aim of this paper is to introduce a generalization of the chordal class for the unit ball $\mathbb{B}_n$.

[1]There is no standard convention in the literature and often $-G$ is called the infinitesimal generator of the semigroup.

[2] Note that there is no standard use of the words "radial" and "chordal" in the literature. In [CDMG10], e.g., an element $G \in \mathrm{Inf}(\mathbb{D})$ is called *radial* if $\tau \in \mathbb{D}$ and chordal if $\tau \in \partial\mathbb{D}$.

1.2 The hydrodynamic normalization in one dimension

Instead of fixing an interior point, like in the class S, it can be of interest to investigate univalent self-mappings of $\mathbb{D}$ that fix a boundary point. In this case, one usually passes from $\mathbb{D}$ to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$.
A class of such mappings that is easy to describe and that appears in several applications is the set of all univalent mappings $f : \mathbb{H} \to \mathbb{H}$ that fix the boundary point ∞ and have the so called *hydrodynamic normalization.* Basic properties of this class can be found in [GB92], see also [Bau05] and [CDMG10]. One of its main applications is the chordal Loewner equation, see [ABCDM10], Section 4, for further references.

A univalent function $f : \mathbb{H} \to \mathbb{H}$ has *hydrodynamic normalization* (at ∞) if f has the expansion

$$f(z) = z - \frac{c}{z} + \gamma(z),$$

where $c \geq 0$, which is usually called *half-plane capacity,* and γ satisfies $\angle \lim_{z\to\infty} z \cdot \gamma(z) = 0$.
We denote by $\mathfrak{P}$ the set of all these functions. Then $\mathfrak{P}$ is a semigroup and the functional $l : \mathfrak{P} \to [0, \infty)$, $l(f) = c$, is additive: If $f_1, f_2 \in \mathfrak{P}$, then $f_1 \circ f_2 \in \mathfrak{P}$ and $l(f_1 \circ f_2) = l(f_1) + l(f_2)$.

Remark 1.3. *Let $f \in \mathfrak{P}$ with $l(f) = c$. If we transfer f to the unit disc by conjugation by the Cayley transform, then we obtain a function $\tilde{f} : \mathbb{D} \to \mathbb{D}$ having the expansion*

$$\tilde{f}(z) = z - \frac{c}{4}(z - 1)^3 + \tilde{\gamma}(z),$$

where $\angle \lim_{z\to 1} \frac{\tilde{\gamma}(z)}{(z-1)^3} = 0$.

If $\{\Phi_t\}_{t\geq 0}$ is a one-real-parameter semigroup contained in $\mathfrak{P}$ with $l(\Phi_1) = a$, then it is easy to see that $l(\Phi_t) = a \cdot t$. If H is the generator of this semigroup, then we also define $l(H) := a$.
We will be interested in the following set of chordal generators.

Definition 1.4. By $\mathcal{K}(\mathbb{H}, c)$ we denote the set of all infinitesimal generators H of one-real parameter semigroups $\{\Phi_t\}_{t\geq 0}$ contained in $\mathfrak{P}$ with $l(H) \leq c$.

Remark 1.5. *The set $\mathcal{K}(\mathbb{H}, c)$ can be characterized in various ways; see [GB92], Section 1 and [Maa92], Proposition 2.2.*
It is known that $H \in \mathcal{K}(\mathbb{H}, c)$ for some $c \geq 0$ if and only if H maps $\mathbb{H}$ into $\overline{\mathbb{H}}$ and

$$\limsup_{y\to\infty} y|H(iy)| \leq c. \tag{1.3}$$

In fact, $l(H) = \limsup_{y\to\infty} y|H(iy)|$.
Furthermore, this is equivalent to: H maps $\mathbb{H}$ into $\overline{\mathbb{H}}$ and

$$|H(z)| \leq \frac{c}{Im(z)} \tag{1.4}$$

for all $z \in \mathbb{H}$. The number $l(H)$ is the smallest constant such that this inequality holds.
Finally, it is known that this property is equivalent to the fact that $-G$ is the Cauchy transform of a finite, non-negative Borel measure μ on $\mathbb{R}$, i.e.

$$H(z) = \int_{\mathbb{R}} \frac{\mu(du)}{u - z}. \tag{1.5}$$

The number $l(H)$ can be calculated by $l(H) = \mu(\mathbb{R})$.

Remark 1.6. *It is easy to see that the following holds: if $f \in \mathfrak{P}$ with $c = l(f)$, then $H := f - id \in \mathcal{K}(\mathbb{H}, c)$ with $l(H) = c$.*

Let $C : \mathbb{H} \to \mathbb{D}$, $C(z) = \frac{z-i}{z+i}$, be the Cayley map. We define $\mathcal{K}(\mathbb{D}, c)$ by

$$\mathcal{K}(\mathbb{D}, c) = \{C'(C^{-1}) \cdot (H \circ C^{-1}) \,|\, H \in \mathcal{K}(\mathbb{H}, c)\}.[3]$$

The rest of this paper is organized as follows: In Section 2 we look for an invariant characterization of chordal generators, i.e. of the sets $\mathcal{K}(\mathbb{H}, c)$ and $\mathcal{K}(\mathbb{D}, c)$, and we introduce the class $\mathcal{K}(\mathbb{B}_n, c)$ for the higher dimensional unit ball. It will turn out to be quite useful to study "slices" of this class, which is done in Section 3. In Section 4 we introduce and study the class $\mathfrak{P}_n$, a higher dimensional analog of the class $\mathfrak{P}$.

[3] If $\{\Phi_t\}_{t\geq 0}$ is a semigroup in $\mathbb{H}$ with generator H, then $\{C \circ \Phi_t \circ C^{-1}\}_{t\geq 0}$ is a semigroup in $\mathbb{D}$ and its generator is given by $C'(C^{-1}) \cdot (H \circ C^{-1})$.

2 Chordal generators in higher dimensions

2.1 Invariant formulation for $\mathcal{K}(\mathbb{D}, c)$ and $\mathcal{K}(\mathbb{H}, c)$

For $R > 0$ we let $E_{\mathbb{D}}(1, R)$ be the horodisc in $\mathbb{D}$ with center 1 and radius R, i.e.

$$E_{\mathbb{D}}(1, R) = \left\{ z \in \mathbb{D} \,|\, \frac{1}{|u_{\mathbb{D}}(z)|} < R \right\},$$

where $u_{\mathbb{D}}(z) = -\frac{1-|z|^2}{|1-z|^2}$ is the Poisson kernel in $\mathbb{D}$ with respect to 1.
By using the Cayley map we define analogously $E_{\mathbb{H}}(\infty, R) = C^{-1}(E_{\mathbb{D}}(1, R)) = \{z \in \mathbb{H} \,|\, \frac{1}{\mathrm{Im}(z)} < R\}$.
For $z \in \mathbb{D}$ and a tangent vector $v \in \mathbb{C}$ we denote by $|v|_{\mathbb{D},z}$ the hyperbolic length of v (with curvature -1), i.e.

$$|v|_{\mathbb{D},z} := \frac{2|v|}{1 - |z|^2}.$$

Furthermore, we let $R_{\mathbb{D}}(z)$ be the radius R of the horodisc $E_{\mathbb{D}}(1, R)$ that satisfies $z \in \partial E(1, R)$; in short $R_{\mathbb{D}}(z) = \frac{1}{|u_{\mathbb{D}}(z)|}$. Analogously, for $z \in \mathbb{H}$ and $v \in \mathbb{C}$, we define $R_{\mathbb{H}}(z) := 1/\mathrm{Im}(z)$ and the hyperbolic length $|v|_{\mathbb{H},z} := |v|/\mathrm{Im}(z)$.

According to (1.4) we know that $H \in \mathcal{K}(\mathbb{H}, c)$ if and only if H maps $\mathbb{H}$ into $\overline{\mathbb{H}}$ and $|H(z)| \leq c/\mathrm{Im}(z)$ for all $z \in \mathbb{H}$. By using the Berkson-Porta formula it is easy to see that we can rephrase this to: $H \in \mathcal{K}(\mathbb{H}, c)$ if and only if $H \in \mathrm{Inf}(\mathbb{H})$ and $|H(z)| \leq c/\mathrm{Im}(z)$ for all $z \in \mathbb{H}$.
The last inequality is equivalent to $|H(z)|/\mathrm{Im}(z) \leq c/\mathrm{Im}(z)^2$ or

$$|H(z)|_{\mathbb{H},z} \leq \frac{c}{\mathrm{Im}(z)^2} = c \cdot R_{\mathbb{H}}(z)^2.$$

If we pass from $\mathbb{H}$ to $\mathbb{D}$ and transform H into $G = C'(C^{-1}) \cdot (H \circ C^{-1})$, then G satisfies $|G(C(z))|_{\mathbb{D},C(z)} = |H(z)|_{\mathbb{H},z}$ and we immediately get the following characterization.

Proposition 2.1. Let $G \in \mathrm{Inf}(\mathbb{D})$. Then

$$G \in \mathcal{K}(\mathbb{D}, c) \quad \iff \quad |G(z)|_{\mathbb{D},z} \leq c \cdot R_{\mathbb{D}}(z)^2 \quad \text{for all} \quad z \in \mathbb{D}.$$

Let $H \in \mathrm{Inf}(\mathbb{H})$. Then

$$H \in \mathcal{K}(\mathbb{H}, c) \quad \iff \quad |H(z)|_{\mathbb{H},z} \leq c \cdot R_{\mathbb{H}}(z)^2 \quad \text{for all} \quad z \in \mathbb{H}.$$

2.2 Chordal generators in the unit ball

For $n \in \mathbb{N}$, let u_n be the pluricomplex Poisson kernel in $\mathbb{B}_n$ with pole at $e_1 := (1, 0, ..., 0)$, i.e.

$$u_{\mathbb{B}_n,p} = -\frac{1 - \|z\|^2}{|1 - z_1|^2}.$$

The level sets of $u_{\mathbb{B}_n}$ are exactly the boundaries of horospheres with center e_1, more precisely, the set

$$E_{\mathbb{B}_n}(e_1, R) := \{z \in \mathbb{B}_n \,|\, |u_{\mathbb{B}_n}(z)|^{-1} < R\}, R > 0,$$

is the horosphere with center e_1 and radius R.
Furthermore, for $z \in \mathbb{B}_n$ and $v \in \mathbb{C}^n$ we denote by $\|v\|_{\mathbb{B}_n,z}$ the Kobayashi-hyperbolic length of the vector v with respect to z.
Motivated by Proposition 2.1, we make the following definition.

Definition 2.2. Let $c \geq 0$. We define the class $\mathcal{K}(\mathbb{B}_n, c)$ to be the set of all infinitesimal generators G on $\mathbb{B}_n$ such that for all $z \in \mathbb{B}_n$ the following inequality holds:

$$\|G(z)\|_{\mathbb{B}_n,z} \leq \frac{c}{u_{\mathbb{B}_n}(z)^2}. \tag{2.1}$$

Remark 2.3. *$\mathcal{K}(\mathbb{B}_n, c)$ is a compact family: Montel's theorem and the definition of $\mathcal{K}(\mathbb{B}_n, c)$ immediately imply that it is a normal family. If a sequence $(G_n) \subset \mathcal{K}(\mathbb{B}_n, c)$ converges locally uniformly to $G : \mathbb{B}_n \to \mathbb{C}^n$, then G is holomorphic and also an infinitesimal generator which can be seen by using the characterization given in [BCDM10], Theorem 0.2. Of course, G also satisfies* (2.1) *and we conclude $G \in \mathcal{K}(\mathbb{B}_n, c)$.*

Just as we passed from $\mathbb{D}$ to $\mathbb{H}$ in one dimension, we can pass from the unit ball $\mathbb{B}_n$ to the Siegel upper half-space $\mathbb{H}_n = \{(z_1, \tilde{z}) \in \mathbb{C}^n \,|\, \mathrm{Im}(z_1) > \|\tilde{z}\|^2\}$ in order to get simpler formulas:
The Cayley map

$$C : \mathbb{H}_n \to \mathbb{B}_n, \quad C(z) = (C_1(z), ..., C_n(z)) = \left(\frac{z_1 - i}{z_1 + i}, \frac{2z_2}{z_1 + i}, ..., \frac{2z_n}{z_1 + i}\right),$$

maps $\mathbb{H}_n$ biholomorphically onto $\mathbb{B}_n$. It extends to a homeomorphism from the one-point compactification $\widehat{\mathbb{H}}_n = \mathbb{H}_n \cup \partial\mathbb{H}_n \cup \{\infty\}$ of $\mathbb{H}_n \cup \partial\mathbb{H}_n$ to the closure of $\mathbb{B}^n$.
The pluricomplex Poisson kernel transforms as follows:

$$u_{\mathbb{H}_n}(z) := u_{\mathbb{B}_n}(C(z)) = -\mathrm{Im}(z_1) + \|\tilde{z}\|^2.$$

Thus, we define the horosphere $E_{\mathbb{H}_n}(\infty, R)$ with center ∞ and radius $R > 0$ by

$$E_{\mathbb{H}_n}(\infty, R) := \{z \in \mathbb{H}_n \,|\, \mathrm{Im}(z_1) - \|\tilde{z}\|^2 > \frac{1}{R}\}.$$

For $v \in \mathbb{C}^n$ and $z \in \mathbb{H}_n$ we let $\|v\|_{\mathbb{H}_n,z}$ be the Kobayashi hyperbolic length of v.

Let $c \geq 0$. We define the class $\mathcal{K}(\mathbb{H}_n, c)$ to be the set of all infinitesimal generators H on $\mathbb{H}_n$ satisfying the inequality

$$\|H(z)\|_{\mathbb{H}_n,z} \leq \frac{c}{u_{\mathbb{H}_n}(z)^2}$$

for all $z \in \mathbb{H}_n$. Then we have

$$\mathcal{K}(\mathbb{B}_n, c) = \{C'(C^{-1}) \cdot (H \circ C^{-1}) \,|\, H \in \mathcal{K}(\mathbb{H}_n, c)\}.$$

From now on we will stay in the upper half-space $\mathbb{H}_n$, where most of the computations we need take a simpler form.

3 Slices

3.1 Normalized geodesics and slices

For any $H \in \mathrm{Inf}(\mathbb{H}_n)$ one can consider one-dimensional slices by using the so called *Lempert projection devices*; see [BS14], Section 3.
If $w \in \mathbb{H}_n$, then there exists a unique complex passing through w and ∞. Let us choose a parametrization $\varphi : \mathbb{H} \to \mathbb{H}_n$ of this geodesic. There exists a unique holomorphic map $P : \mathbb{H}_n \to \mathbb{H}_n$ with $P^2 = P$ and $P \circ \varphi = \varphi$. Define $\tilde{P} = \varphi^{-1} \circ P$. Then

$$h_\varphi : \mathbb{H} \to \mathbb{C}, \qquad h_\varphi(\zeta) = d\tilde{P}(\varphi(\zeta)) \cdot H(\varphi(\zeta)),$$

is an infinitesimal generator on $\mathbb{H}$; see [BS14], p. 6.
We will need special parametrizations of these geodesics: In [BP05], p. 516, it is shown that for any complex geodesic $\varphi : \mathbb{H} \to \mathbb{H}_n$ with $\varphi(\infty) = \infty$, there exists $a_\varphi > 0$ such that

$$u_{\mathbb{H}_n}(\varphi(\zeta)) = a_\varphi \cdot u_{\mathbb{H}}(\zeta)$$

for all $\zeta \in \mathbb{H}$. Call a geodesic $\varphi : \mathbb{H} \to \mathbb{H}_n$ *normalized* if $\varphi(\infty) = \infty$ and $a_\varphi = 1$.

Lemma 3.1. *Let $a \in \mathbb{C}$ and $\gamma \in \mathbb{C}^{n-1}$ such that $(a, \gamma) \in \mathbb{H}_n$. Then the map*

$$\varphi_\gamma : \mathbb{H} \to \mathbb{H}_n, \quad \varphi_\gamma(\zeta) := (\zeta + i\|\gamma\|^2, \gamma),$$

is a normalized geodesic through (a, γ). Furthermore, if $H = (H_1, \tilde{H}) \in Inf(\mathbb{H}_n)$, then the slice $h_\gamma := h_{\varphi_\gamma}$ of H with respect to φ_γ is given by

$$h_\gamma(\zeta) = H_1(\varphi_\gamma(\zeta)) - 2i\overline{\gamma}^T \cdot \tilde{H}(\varphi_\gamma(\zeta)). \tag{3.1}$$

Proof. Let $\psi : \mathbb{D} \to \mathbb{B}_n$ be a complex geodesic with $\psi(1) = e_1$. As a parametrization for ψ one can choose (see Section 3 in [BS14]) $\psi(\zeta) = (\alpha^2(\zeta - 1) + 1, \alpha(\zeta - 1)\beta)$, where $\alpha > 0$ and $\beta \in \mathbb{C}^{n-1}$ such that $\|\beta\|^2 = 1 - \alpha^2$. Then $C^{-1}(\psi(\zeta)) = (i\frac{2+\alpha^2(\zeta-1)}{\alpha^2(1-\zeta)}, i\beta/\alpha)$ and

$$\zeta \mapsto C^{-1}(\psi(C_1(\zeta))) = (-i + \frac{\zeta + i}{\alpha^2}, i\beta/\alpha) = (\frac{\zeta}{\alpha^2} + i\frac{1 - \alpha^2}{\alpha^2}, i\beta/\alpha) = (\frac{\zeta}{\alpha^2} + i\left\|\frac{\beta}{\alpha}\right\|^2, i\beta/\alpha)$$

is a complex geodesic from $\mathbb{H}$ to $\mathbb{H}_n$. A reparametrization $[\zeta/\alpha^2$ to $\zeta]$ and setting $\gamma = i\beta/\alpha$ gives the geodesic

$$\varphi_\gamma(\zeta) = (\zeta + i\|\gamma\|^2, \gamma). \tag{3.2}$$

This complex geodesic is normalized because it satisfies $\varphi_\gamma(\infty) = \infty$ and

$$u_{\mathbb{H}_n}(\varphi_\gamma(\zeta)) = \mathrm{Im}(\zeta + i\|\gamma\|^2) - \|\gamma\|^2 = \mathrm{Im}(\zeta) = u_{\mathbb{H}}(\zeta).$$

The projection onto $\varphi_\gamma(\mathbb{H})$ is given by

$$P(z_1, \tilde{z}) = (z_1 - 2i\overline{\gamma}^T \cdot \tilde{z} + 2i\|\gamma\|^2, \gamma). \tag{3.3}$$

Clearly, P is holomorphic and maps $\mathbb{H}_n$ onto $\varphi_\gamma(\mathbb{H})$ because

$$\begin{aligned} \mathrm{Im}(z_1 - 2i\overline{\gamma}^T \cdot \tilde{z} + 2i\|\gamma\|^2) &= \mathrm{Im}(z_1) - 2\mathrm{Im}(i\overline{\gamma}^T \cdot \tilde{z}) + 2\|\gamma\|^2 \\ &\geq \|\tilde{z}\|^2 - 2\|\gamma\|\|\tilde{z}\| + \|\gamma\|^2 + \|\gamma\|^2 = (\|\gamma\| - \|\tilde{z}\|)^2 + \|\gamma\|^2 \geq \|\gamma\|^2. \end{aligned}$$

Furthermore,

$$(P \circ P)(z_1, \tilde{z}) = (z_1 - 2i\overline{\gamma}^T\tilde{z} + 2i\|\gamma\|^2 - 2i\overline{\gamma}^T\gamma + 2i\|\gamma\|^2, \gamma) = (z_1 - 2i\overline{\gamma}^T\tilde{z} + 2i\|\gamma\|^2, \gamma) = P(z_1, \tilde{z}).$$

Thus, the inverse $\tilde{P} : \mathbb{H}_2 \to \mathbb{H}, \tilde{P} = \varphi_\gamma^{-1} \circ P$, is given by $\tilde{P}(z_1, \tilde{z}) = (z_1 - 2i\overline{\gamma}^T\tilde{z} + i\|\gamma\|^2)$.
If $H(z) = (H_1(z), \tilde{H}(z))$ is a generator on $\mathbb{H}_n$, we get the slice reduction

$$h_{\varphi_\gamma}(\zeta) = d\tilde{P}(\varphi_\gamma(\zeta)) \cdot H(\varphi_\gamma(\zeta)) = H_1(\varphi_\gamma(\zeta)) - 2i\overline{\gamma}^T \cdot \tilde{H}(\varphi_\gamma(\zeta)).$$

□

3.2 Some explicit formulas

Later on we will need explicit formulas of the Kobayashi norms of $dP(z)H(z)$ and $H(z) - dP(z) \cdot H(z)$. The following lemma is proven in the Appendix.

Lemma 3.2. *Let $a \in \mathbb{C}, p, v \in \mathbb{C}^{n-1}$ and $z = (z_1, \tilde{z}) \in \mathbb{H}_n$. Then the following formulas hold:*

$$\left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\|_{\mathbb{H}_n, z} = \frac{|a|}{|u_{\mathbb{H}_n}(z)|}, \tag{3.4}$$

$$\left\| \begin{pmatrix} 2i\overline{p}^T v \\ v \end{pmatrix} \right\|_{\mathbb{H}_n, z} = 2\frac{\sqrt{\|v\|^2|u_{\mathbb{H}_n}(z)| + |\overline{(p-\tilde{z})}^T v|^2}}{|u_{\mathbb{H}_n}(z)|}, \tag{3.5}$$

$$\left\| \begin{pmatrix} a - 2i\overline{\tilde{z}}^T v \\ 0 \end{pmatrix} + \begin{pmatrix} 2i\overline{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n, z} = \left\| \begin{pmatrix} a - 2i\overline{\tilde{z}}^T v \\ 0 \end{pmatrix} \right\|^2_{\mathbb{H}_n, z} + \left\| \begin{pmatrix} 2i\overline{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n, z}. \tag{3.6}$$

By using Lemma 3.2 we obtain the following explicit expressions.

Lemma 3.3. *Let $H = (H_1, \tilde{H}) \in Inf(\mathbb{H}_n)$ and fix $z \in \mathbb{H}_n$. Denote by P the projection onto the complex geodesic through z and ∞. Then the following formulas hold:*

$$dP(z) \cdot H(z) = (H_1(z) - 2i\overline{\tilde{z}}^T\tilde{H}(z), 0), \qquad H(z) - dP(z) \cdot H(z) = (2i\overline{\tilde{z}}^T\tilde{H}(z), \tilde{H}(z)). \tag{3.7}$$

Furthermore,

$$\|H(z)\|^2_{\mathbb{H}_n, z} = \|dP(z) \cdot H(z)\|^2_{\mathbb{H}_n, z} + \|H(z) - dP(z) \cdot H(z)\|^2_{\mathbb{H}_n, z}, \tag{3.8}$$

$$\|dP(z)H(z)\|_{\mathbb{H}_n, z} = \frac{|H_1(z) - 2i\overline{\tilde{z}}^T\tilde{H}(z)|}{|u_{\mathbb{H}_n}(z)|}, \tag{3.9}$$

$$\|H(z) - dP(z) \cdot H(z)\|_{\mathbb{H}_n, z} = 2\frac{\|\tilde{H}(z)\|}{\sqrt{|u_{\mathbb{H}_n}(z)|}}. \tag{3.10}$$

Proof. The formulas for $dP(z)H(z)$ and $H(z) - dP(z)H(z)$ follow from the explicit form (3.3). Equation (3.8) follows from (3.6) with $a = H_1(z)$ and $v = \tilde{H}(z)$.
Furthermore, equation (3.9) follows directly from (3.4) with $a = H_1(z) - 2i\overline{\tilde{z}}^T\tilde{H}(z)$ and equation (3.10) from (3.5) by setting $p = \tilde{z}$ and $v = \tilde{H}$. □

3.3 Slices of generators in $\mathcal{K}(\mathbb{H}_n, c)$ and examples

Proposition 3.4. Let $c \geq 0$ and $H \in \mathcal{K}(\mathbb{H}_n, c)$. Then every normalized slice h_γ of H belongs to $\mathcal{K}(\mathbb{H}, c)$.

Proof. Fix $\gamma \in \mathbb{C}^{n-1}$ and $\zeta \in \mathbb{H}$ and let $z = \varphi_\gamma(\zeta)$.
Furthermore, let P be the projection onto $\varphi_\gamma(\mathbb{H})$. Now we write $H(z)$ as

$$H(z) = dP(z) \cdot H(z) + (H(z) - dP(z)H(z)).$$

As $H \in \mathcal{K}(\mathbb{H}_n, c)$, equation (3.8) implies

$$\|H(z)\|^2_{\mathbb{H}_n,z} = \|dP(z) \cdot H(z)\|^2_{\mathbb{H}_n,z} + \|H(z) - dP(z)H(z)\|^2_{\mathbb{H}_n,z} \leq \frac{c^2}{u_{\mathbb{H}_n}(z)^4}.$$

In particular,

$$\|dP(z) \cdot H(z)\|_{\mathbb{H}_n,z} \leq \frac{c}{u_{\mathbb{H}_n}(z)^2}. \tag{3.11}$$

By the definition of the slice h_γ we have

$$dP(\varphi_\gamma(\zeta)) \cdot H(\varphi_\gamma(\zeta)) = (d\varphi_\gamma)(\zeta) \cdot h_\gamma(\zeta)$$

and consequently

$$\|dP(\varphi_\gamma(\zeta)) \cdot H(\varphi_\gamma(z))\|_{\mathbb{H}_n,\varphi_\gamma(\zeta)} = \|(d\varphi_\gamma)(\zeta) \cdot h_\gamma(\zeta)\|_{\mathbb{H}_n,\varphi_\gamma(\zeta)} = |h_\gamma(\zeta)|_{\mathbb{H},\zeta}.$$

The last equality holds as φ_γ is a complex geodesic. Equation (3.11) implies

$$|h_\gamma(\zeta)|_{\mathbb{H},\zeta} \leq \frac{c}{u_{\mathbb{H}_n}(\varphi_\gamma(\zeta))^2} = \frac{c}{u_{\mathbb{H}}(\zeta)^2},$$

where the last equality holds as φ_γ is normalized. Hence, $h_\gamma \in \mathcal{K}(\mathbb{H}, c)$. □

Remark 3.5. *If two holomorphic functions $H_1, H_2 : \mathbb{H}_n \to \mathbb{C}^n$ have the same slices, i.e. $dP(z)H_1(z) = dP(z)H_2(z)$ for all $z \in \mathbb{H}_n$, then $H_1 = H_2$; see the proof of Theorem 3.2 in [Cas10].*

Example 3.6. The family $\{\Phi_t(z) = (z_1, e^{-it/z_1}z_2)\}_{t\geq 0}$ is a semigroup on $\mathbb{H}_2$. Its generator H is given by

$$H(z_1, z_2) = (0, -i\frac{z_2}{z_1}).$$

Thus, for $\gamma \in \mathbb{C}$ the slice h_γ has the form

$$h_\gamma(z) = -2i\overline{\gamma} \cdot -i\frac{\gamma}{z + i|\gamma|^2} = \frac{-2|\gamma|^2}{z + i|\gamma|^2}.$$

Consequently, the limit $\lim_{y\to\infty} y \cdot |h(iy)| = 2|\gamma|^2$ exists, but does not have an upper bound that is independent of γ. Proposition 3.4 implies that for any $c \geq 0$, $H \notin \mathcal{K}(\mathbb{H}_2, c)$. ★

Example 3.7. Let

$$H : \mathbb{H}_2 \to \mathbb{C}^2, \qquad H(z_1, z_2) = \begin{pmatrix} \frac{-1}{z_1} \\ \frac{z_2}{2z_1^2} \end{pmatrix}.$$

For $\gamma \in \mathbb{C}$ the slice h_γ is given by

$$h_\gamma(\zeta) = \frac{-1}{\zeta + i|\gamma|^2} - 2i\overline{\gamma} \cdot \frac{\gamma}{2(\zeta + i|\gamma|^2)^2} = \frac{-\zeta - 2i|\gamma|^2}{(\zeta + i|\gamma|^2)^2} = \frac{(-\zeta - 2i|\gamma|^2)(\overline{\zeta}^2 - 2i|\gamma|^2\overline{\zeta} - |\gamma|^4)}{|\zeta + i|\gamma|^2|^4}.$$

Let us write $\zeta = x + iy$; $x \in \mathbb{R}, y \in (0, \infty)$. Then a small calculation gives

$$\operatorname{Im}(h_\gamma(\zeta)) = \frac{y(x^2 + y^2) + 4y^2|\gamma|^2 + 5y|\gamma|^4 + 2|\gamma|^6}{|\zeta + i|\gamma|^2|^4} > 0.$$

Furthermore,

$$\limsup_{y\to\infty} y|h_\gamma(iy)| = 1.$$

Hence, $h_\gamma \in \mathcal{K}(\mathbb{H}, 1)$. So each slice is an infinitesimal generator in $\mathbb{H}$ and by [BS14], Proposition 3.8, the function H is an infinitesimal generator in $\mathbb{H}_2$.
Now let $(z_1, z_2) \in \mathbb{H}_2$ and write $z_1 = x + iy$, $x, y \in \mathbb{R}$. Then we get (an explicit formula of the Kobayashi metric is given in the appendix)

$$u_{\mathbb{H}_2}(z)^4 \cdot \|H(z)\|^2_{\mathbb{H}_2,z} = (y - |z_2|^2)^2 \cdot \frac{x^2 + y^2 + 3|z_2|^2 y}{(x^2 + y^2)^2} \underset{y \geq |z_2|^2}{\leq} y^2 \cdot \frac{x^2 + y^2 + 3y^2}{(x^2 + y^2)^2} \leq \frac{x^2 + 4y^2}{x^2 + y^2} \leq 4.$$

Consequently, $H \in \mathcal{K}(\mathbb{H}_2, 2)$. ★

Question 3.8. *Let $H : \mathbb{H}_n \to \mathbb{C}^n$ be an infinitesimal generator. Assume there exists $c \geq 0$ such that $h_\gamma \in \mathcal{K}(\mathbb{H}, c)$ for every $\gamma \in \mathbb{C}^{n-1}$. Does this imply that $H \in \mathcal{K}(\mathbb{H}_n, C)$ for some $C \geq c$?*

4 Univalent functions with hydrodynamic normalization

Motivated by Remark 1.6 we define the following generalization of the class $\mathfrak{P}$, where *id* stands for the identity mapping on $\mathbb{H}_n$.

Definition 4.1.

$$\mathfrak{P}_n := \{f : \mathbb{H}_n \to \mathbb{H}_n \,|\, f \text{ is univalent and } f - id \in \mathcal{K}(\mathbb{H}_n, c) \text{ for some } c \geq 0\}.$$

Remark 4.2. *It is important to note that if $f : \mathbb{H}_n \to \mathbb{H}_n$ is a holomorphic self-mapping, then the map $f - id$ is automatically an infinitesimal generator; see [RS05], p. 207.*

4.1 Basic properties of $\mathfrak{P}_n$

The following proposition summarizes some basic properties of $\mathfrak{P}_n$.

Proposition 4.3.

a) $\mathfrak{P}_n$ contains no automorphism of $\mathbb{H}_n$ except the identity.
b) Let $\alpha : \mathbb{H}_n \to \mathbb{H}_n$ be an automorphism of $\mathbb{H}_n$ with $\alpha(\infty) = \infty$. If $f \in \mathfrak{P}_n$, then $\alpha^{-1} \circ f \circ \alpha \in \mathfrak{P}_n$.
c) Let $f \in \mathfrak{P}_n$. Then $f(E_{\mathbb{H}_n}(\infty, R)) \subset E_{\mathbb{H}_n}(\infty, R)$ for every $R > 0$.
d) Let $f \in \mathfrak{P}_n$ and write $f(z) = z + H(z)$ with $H = (H_1, \tilde{H}) \in \mathcal{K}(\mathbb{H}_n, c)$. Then

$$\|\tilde{H}(z)\|^2 \leq |H_1(z) - 2i\overline{\tilde{z}}^T \tilde{H}| \text{ for all } z = (z_1, \tilde{z}) \in \mathbb{H}_n. \tag{4.1}$$

e) Let $f \in \mathfrak{P}_n$. Then there exists $R > 0$ such that $E_{\mathbb{H}_n}(\infty, R) \subset f(\mathbb{H}_n)$.

Proof. The statements a) and b) can easily be shown by using the explicit form of automorphisms of $\mathbb{H}_n$; see Proposition 2.2.4 in [Aba89].
The statement c) is just Julia's lemma: Write $f(z) = z + H(z)$ and let us pass to the unit ball and define $\tilde{f} : \mathbb{B}_n \to \mathbb{B}_n$, $\tilde{f} = C \circ f \circ C^{-1}$. Then

$$\tilde{f} = \frac{1}{2i + H_1(C^{-1}(z)) - z_1 H_1(C^{-1}(z))} \left[\begin{pmatrix} (1 - z_1) H_1(C^{-1}(z)) \\ 2(1 - z_1)\tilde{H}(C^{-1}(z)) \end{pmatrix} + 2iz \right].$$

By taking the sequence $z_n = (1 - 1/n, 0)$ it is easy to see that

$$\lim_{n\to\infty} \tilde{f}(z_n) = e_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1 - \|\tilde{f}(z_n)\|}{1 - \|z_n\|} = 1,$$

i.e. e_1 is a boundary regular fixed point of $\tilde{f}$ with boundary dilatation coefficient ≤ 1. Julia's lemma (see Theorem 2.2.21 in [Aba89]) implies that $\tilde{f}(E_{\mathbb{B}_n}(e_1, R)) \subset E_{\mathbb{B}_n}(e_1, R)$ for any $R > 0$.
Inequality d) follows directly from c): Let $z = (z_1, \tilde{z}) \in \mathbb{H}_n$. Another formulation of c) is $-u_{\mathbb{H}_n}(z + H(z)) \geq -u_{\mathbb{H}_n}(z)$, or more explicitly

$$\begin{aligned} & \mathrm{Im}(z_1) + \mathrm{Im}(H_1(z)) - \|\tilde{z} + \tilde{H}(z)\|^2 \geq \mathrm{Im}(z_1) - \|\tilde{z}\|^2 \\ \iff \quad & \mathrm{Im}(H_1(z)) \geq \|\tilde{z} + \tilde{H}(z)\|^2 - \|\tilde{z}\|^2 = 2\mathrm{Re}(\overline{\tilde{z}}^T \tilde{H}(z)) + \|\tilde{H}(z)\|^2 \\ \iff \quad & \mathrm{Im}(H_1(z) - 2i\overline{\tilde{z}}^T \tilde{H}(z)) \geq \|\tilde{H}(z)\|^2. \end{aligned}$$

From this inequality it follows that $\|\tilde{H}(z)\|^2 \leq |H_1(z) - 2i\overline{\tilde{z}}^T \tilde{H}|$ for all $z \in \mathbb{H}_n$.

Finally we prove e):
Let $f \in \mathfrak{P}_n$ and write $f(z) = z + H(z)$ with $H \in \mathcal{K}(\mathbb{H}_n, c)$. Because of c) f maps the horosphere $E_{\mathbb{H}_n}(\infty, 1)$ into itself. Hence the statement is proven if we can show that $u_{\mathbb{H}_n}$ is bounded on $f(\partial E_{\mathbb{H}_n}(\infty, 1))$.
Let $z \in \mathbb{H}_n$ with $z \in \partial E_{\mathbb{H}_n}(\infty, 1)$, i.e. $|u_{\mathbb{H}_n}(z)| = 1$. Furthermore, we choose $\zeta \in \mathbb{H}$ and $\gamma \in \mathbb{C}$ such that $\varphi_\gamma(\zeta) = z$. Note that this implies $|u_{\mathbb{H}}(\zeta)| = \mathrm{Im}(\zeta) = 1$.
Let P be the projection onto $\varphi_\gamma(\mathbb{H})$.

Then we have $|u_{\mathbb{H}_n}(f(z))| = |u_{\mathbb{H}_n}(z + H(z))| = |u_{\mathbb{H}_n}(\underbrace{z + dP(z)H(z)}_{=:w} + \underbrace{H(z) - dP(z)H(z)}_{=:v})|$. As $dP(z) \cdot dP(z) = dP(z)$, we have $dP(z) \cdot v = 0$. A small calculation (see also Lemma 3.1 in [Cas10]) gives $v \in T_z^{\mathbb{C}} \partial E_{\mathbb{H}_n}(\infty, 1)$. Furthermore, also $w \in \varphi_\gamma(\mathbb{H})$ and $dP(z) = dP(w)$ and we get $v \in T_w^{\mathbb{C}} \partial E_{\mathbb{H}_n}(\infty, |u_{\mathbb{H}_n}(w)|^{-1})$. As $E_{\mathbb{H}_n}(\infty, |u_{\mathbb{H}_n}(w)|^{-1}) = \{z \in \mathbb{H}_n \,|\, |u_{\mathbb{H}_n}(z)| > |u_{\mathbb{H}_n}(w)|\}$ is convex this implies

$$\begin{aligned} & |u_{\mathbb{H}_n}(w + v)| \leq |u_{\mathbb{H}_n}(w)| = |u_{\mathbb{H}_n}(z + dP(z)H(z))| \underset{\text{Lemma 3.3}}{=} |u_{\mathbb{H}_n}(z + (h_\gamma(\zeta), 0))| \\ = \; & \mathrm{Im}(z_1) - \|\tilde{z}\|^2 + \mathrm{Im}(h_\gamma(\zeta)) \leq \mathrm{Im}(z_1) - \|\tilde{z}\|^2 + |h_\gamma(\zeta)| \\ = \; & |u_{\mathbb{H}_n}(z)| + |h_\gamma(\zeta)| = 1 + |h_\gamma(\zeta)| \leq 1 + \frac{c}{\mathrm{Im}(\zeta)} = 1 + c. \end{aligned}$$

Consequently, $f(\mathbb{H}_n) \supset f(E_{\mathbb{H}_n}(\infty, 1)) \supset E_{\mathbb{H}_n}(\infty, 1 + c)$. $\square$

Theorem 4.4. *$\mathfrak{P}_n$ is a semigroup: If $f, g \in \mathfrak{P}_n$, then $f \circ g \in \mathfrak{P}_n$.*

Proof. Let $f, g \in \mathfrak{P}_n$ with $F = (F_1, \tilde{F}) := f - id, G = (G_1, \tilde{G}) := g - id$ and

$$\|F(z)\|_{\mathbb{H}_n, z} \leq \frac{c}{u_{\mathbb{H}_n}(z)^2}, \qquad \|G(z)\|_{\mathbb{H}_n, z} \leq \frac{d}{u_{\mathbb{H}_n}(z)^2}$$

for all $z \in \mathbb{H}_n$. Let $z = (z_1, \tilde{z}) \in \mathbb{H}_n$ and $p = (p_1, \tilde{p}) := z + G(z)$.
From Remark 4.2 we know that $f \circ g - id$ is an infinitesimal generator on $\mathbb{H}_n$. It remains to estimate the hyperbolic metric of this generator. We have

$$\begin{aligned} & \|(f \circ g)(z) - z\|_{\mathbb{H}_n, z} = \|G(z) + F(z + G(z))\|_{\mathbb{H}_n, z} \\ \leq \; & \|G(z)\|_{\mathbb{H}_n, z} + \|F(z + G(z))\|_{\mathbb{H}_n, z} \leq \frac{d}{u_{\mathbb{H}_n}(z)^2} + \|F(p)\|_{\mathbb{H}_n, z} \\ \leq \; & \frac{d}{u_{\mathbb{H}_n}(z)^2} + \|(F_1(p) - 2i\overline{\tilde{p}}^T \tilde{F}(p), 0)\|_{\mathbb{H}_n, z} + \|(2i\overline{\tilde{p}}^T \tilde{F}(p), \tilde{F}(p))\|_{\mathbb{H}_n, z}. \end{aligned}$$

Note that $F_1(p) - 2i\overline{\tilde{p}}^T \tilde{F}(p)$ corresponds to the slice of F with respect to the geodesic through p and infinity. Because of Proposition 3.4 we know that

$$|F_1(p) - 2i\overline{\tilde{p}}^T \tilde{F}(p)| \leq \frac{c}{|u_{\mathbb{H}_n}(p)|} \leq \frac{c}{|u_{\mathbb{H}_n}(z)|},$$

where the second inequality follows from Proposition 4.3 c). Together with equation (3.4), this implies

$$\|(F_1(p) - 2i\overline{\tilde{p}}^T \tilde{F}(p), 0)\|_{\mathbb{H}_n, z} = \frac{|(F_1(p) - 2i\overline{\tilde{p}}^T \tilde{F}(p)|}{|u_{\mathbb{H}_n}(z)|} \leq \frac{c}{u_{\mathbb{H}_n}(z)^2}. \tag{4.2}$$

It remains to show that there exists a constant $C > 0$ such that

$$\|(2i\overline{\tilde{p}}^T \tilde{F}(p), \tilde{F}(p))\|_{\mathbb{H}_n, z} \leq \frac{C}{u_{\mathbb{H}_n}(z)^2}.$$

First, equation (3.5) gives

$$\begin{aligned} \|(2i\overline{\tilde{p}}^T \tilde{F}(p), \tilde{F}(p))\|_{\mathbb{H}_n, z} &= 2\frac{\sqrt{\|\tilde{F}(p)\|^2 |u_{\mathbb{H}_n}(z)| + |\overline{(\tilde{p} - \tilde{z})}^T \tilde{F}(p)|^2}}{|u_{\mathbb{H}_n}(z)|} \\ \leq 2\frac{\sqrt{\|\tilde{F}(p)\|^2 |u_{\mathbb{H}_n}(z)| + \|(\tilde{p} - \tilde{z})\|^2 \cdot \|\tilde{F}(p)\|^2}}{|u_{\mathbb{H}_n}(z)|} &= 2\frac{\|\tilde{F}(p)\|}{|u_{\mathbb{H}_n}(z)|}\sqrt{|u_{\mathbb{H}_n}(z)| + \|\tilde{G}(z)\|^2}. \end{aligned} \tag{4.3}$$

Now we differentiate between two cases. Case 1: $|u_{\mathbb{H}_n}(z)| \geq 1$.
The equations (3.8) and (3.10) imply $\frac{2\|\tilde{F}(p)\|}{\sqrt{|u_{\mathbb{H}_n}(p)|}} \leq \|\tilde{F}(p)\|_{\mathbb{H}_n,p} \leq \frac{c}{u_{\mathbb{H}_n}(p)^2}$, thus

$$\|\tilde{F}(p)\| \leq \frac{c}{2|u_{\mathbb{H}_n}(p)|^{3/2}} \leq \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}}. \tag{4.4}$$

In the same way we get

$$\|\tilde{G}(z)\| \leq \frac{d}{2|u_{\mathbb{H}_n}(z)|^{3/2}}. \tag{4.5}$$

Combining (4.4) with (4.3) gives

$$\begin{aligned} \|(2i\overline{\tilde{p}}^T\tilde{F}(p), \tilde{F}(p))\|_{\mathbb{H}_n,z} &\leq \frac{c}{|u_{\mathbb{H}_n}(z)||u_{\mathbb{H}_n}(z)|^{3/2}}\sqrt{|u_{\mathbb{H}_n}(z)| + \|\tilde{G}(z)\|^2} \\ = \frac{c}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{1 + \frac{\|\tilde{G}(z)\|^2}{|u_{\mathbb{H}_n}(z)|}} &\underset{(4.5)}{\leq} \frac{c}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{1 + \frac{d^2}{4|u_{\mathbb{H}_n}(z)|^4}} \leq \frac{c\sqrt{1+\frac{d^2}{4}}}{|u_{\mathbb{H}_n}(z)|^2}. \end{aligned}$$

Case 2: $|u_{\mathbb{H}_n}(z)| \leq 1$.
From equation (4.2) we know that $|F_1(p) - 2i\overline{\tilde{p}}^T\tilde{F}(p)| \leq \frac{c}{|u_{\mathbb{H}_n}(z)|}$ and equation (4.1) implies

$$\|\tilde{F}(p)\| \leq \frac{\sqrt{c}}{\sqrt{|u_{\mathbb{H}_n}(z)|}}.$$

Similarly we get

$$\|\tilde{G}(z)\| \leq \frac{\sqrt{d}}{\sqrt{|u_{\mathbb{H}_n}(z)|}}.$$

Hence, we obtain with (4.3):

$$\begin{aligned} &\|(2i\overline{\tilde{p}}^T\tilde{F}(p), \tilde{F}(p))\|_{\mathbb{H}_n,z} \leq 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^{3/2}}\sqrt{|u_{\mathbb{H}_n}(z)| + \|\tilde{G}(z)\|^2} \\ \leq\ & 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^{3/2}}\sqrt{|u_{\mathbb{H}_n}(z)| + \frac{d}{|u_{\mathbb{H}_n}(z)|}} = 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{u_{\mathbb{H}_n}(z)^2 + d} \\ \leq\ & 2\frac{\sqrt{c}}{|u_{\mathbb{H}_n}(z)|^2}\sqrt{1+d}. \end{aligned}$$

□

4.2 Semigroups with generators in $\mathcal{K}(\mathbb{H}_n, c)$

Theorem 4.5. *Let $\{\Phi_t\}_{t\geq 0}$ be a semigroup on $\mathbb{H}_n$ with generator $H \in \mathcal{K}(\mathbb{H}_n, c)$. Then $\Phi_t \in \mathfrak{P}_n$ for every $t \geq 0$.*

Proof. Firstly, for every $t \geq 0$ and $R > 0$, the map Φ_t maps the horosphere $E_{\mathbb{H}_n}(\infty, R)$ into itself, i.e.

$$|u_{\mathbb{H}_n}(\Phi_t(z))| \geq |u_{\mathbb{H}_n}(z)| \tag{4.6}$$

for every $z \in \mathbb{H}_n$. This can be seen as follows:
Let G be the corresponding generator in the unit ball, i.e. $G = C'(C^{-1}) \cdot (H \circ C^{-1})$. Then G satisfies the inequality

$$\|G(z)\| \leq \|G(z)\|_{\mathbb{B}_n,z} \leq \frac{c}{u_{\mathbb{B}_n}(z)^2} = \frac{c|1-z_1|^4}{(1-\|z\|^2)^2}.$$

Putting $z = r \cdot e_1$ gives

$$\|G(re_1)\| \leq \frac{c(1-r)^4}{(1-r^2)^2} = \frac{c(1-r)^2}{(1+r)^2}.$$

From this it follows immediately that

$$\lim_{(0,1)\ni r\to 1} G(re_1) = 0 \quad \text{and} \quad \lim_{(0,1)\ni r\to 1} \frac{G_1(re_1)}{r-1} = 0.$$

Theorem 0.3 in [BCDM10] implies that e_1 is a boundary regular fixed point for the generated semigroup with boundary dilatation coefficient 1. Hence we can apply Julia's lemma.

Let $z = (z_1, z_2) \in \mathbb{H}_n$ and write $\Phi_t = (\Phi_{1,t}, \tilde{\Phi}_t)$, $H = (H_1, \tilde{H})$. The semigroup Φ_t satisfies the integral equation

$$\Phi_t(z) = z + \int_0^t H(\Phi_s(z))\, ds.$$

Similarly to the proof of Theorem 4.4, equation (4.4), we deduce from the fact that $H \in \mathcal{K}(\mathbb{H}_n, c)$ and equations (3.8) and (3.10) that

$$\|\tilde{H}(\Phi_t(z))\| \leq \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}} \tag{4.7}$$

for every $z \in \mathbb{H}_n$ and $t \geq 0$; and similarly to equation (4.2) we deduce that

$$\|(H_1(\Phi_t(z)) - 2i\overline{\tilde{\Phi}_t}^T \tilde{H}(\Phi_t(z)), 0)\|_{\mathbb{H}_n,z} \leq \frac{c}{u_{\mathbb{H}_n}(z)^2} \tag{4.8}$$

for every $z \in \mathbb{H}_n$ and $t \geq 0$.
First we get

$$\|\tilde{\Phi}_t - \tilde{z}\| \leq \int_0^s \|\tilde{H}(\Phi_\tau(z))\|\, d\tau \leq \int_0^s \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}}\, d\tau = \frac{cs}{2|u_{\mathbb{H}_n}(z)|^{3/2}}. \tag{4.9}$$

Case 1: $|u_{\mathbb{H}_n}(z)| \geq 1$. Then we have:

$$\begin{aligned}
& \|\Phi_t(z) - z\|_{\mathbb{H}_n,z} \leq \int_0^t \|H(\Phi_s(z))\|_{\mathbb{H}_n,z}\, ds \\
\leq\ & \int_0^t \left\| \begin{pmatrix} H_1(\Phi_s(z)) - 2i\overline{\tilde{\Phi}_t}^T \tilde{H}(\Phi_s(z)) \\ 0 \end{pmatrix} \right\|_{\mathbb{H}_n,z} ds + \int_0^t \left\| \begin{pmatrix} 2i\overline{\tilde{\Phi}_t}^T \tilde{H}(\Phi_t(z)), \\ \tilde{H}(\Phi_t(z)), \end{pmatrix} \right\|_{\mathbb{H}_n,z} ds \\
\underset{(4.8),(5.2)}{\leq}\ & \int_0^t \frac{c}{u_{\mathbb{H}_n}(z)^2}\, ds + \int_0^t 2\frac{\|\tilde{H}(\Phi_s(z))\|}{|u_{\mathbb{H}_n}(z)|} \sqrt{|u_{\mathbb{H}_n}(z)| + \|\tilde{\Phi}_t - \tilde{z}\|^2}\, ds \\
\underset{(4.7),(4.9)}{\leq}\ & \int_0^t \frac{c}{u_{\mathbb{H}_n}(z)^2}\, ds + \int_0^t \frac{c}{|u_{\mathbb{H}_n}(z)|^{5/2}} \sqrt{|u_{\mathbb{H}_n}(z)| + \frac{c^2 s^2}{4|u_{\mathbb{H}_n}(z)|^3}}\, ds \\
=\ & \frac{ct}{u_{\mathbb{H}_n}(z)^2} + \int_0^t \frac{c}{|u_{\mathbb{H}_n}(z)|^2} \sqrt{1 + \frac{c^2 s^2}{4|u_{\mathbb{H}_n}(z)|^4}}\, ds \\
\leq\ & \frac{ct}{u_{\mathbb{H}_n}(z)^2} + \int_0^t \frac{c}{|u_{\mathbb{H}_n}(z)|^2} \sqrt{1 + c^2 s^2}\, ds \\
=\ & c \cdot \frac{t + \int_0^t \sqrt{1 + c^2 s^2}\, ds}{u_{\mathbb{H}_n}(z)^2}.
\end{aligned}$$

The case $|u_{\mathbb{H}_n}(z)| \leq 1$ is treated similarly, compare with the proof of Theorem 4.4, and we conclude that for every $t \geq 0$, there exists $C > 0$ such that $\|\Phi_t(z) - z\|_{\mathbb{H}_n} \leq \frac{C}{u_{\mathbb{H}_n}(z)^2}$ for all $z \in \mathbb{H}_n$. Together with Remark 4.2, this implies that $\Phi_t \in \mathfrak{P}_n$. □

Remark 4.6. *Let $H : [0, \infty) \times \mathbb{H}_n \to \mathbb{C}^n$ be a $\mathcal{K}(\mathbb{H}_n, c)-$Herglotz vector field, i.e. $H(t, \cdot) \in \mathcal{K}(\mathbb{H}_n, c)$ for almost every $t \geq 0$ and H satisfies certain regularity conditions, see Definition 1.2 in [AB11]. In this case, one can solve the non-autonomous version of equation* (1.1), *namely the Loewner equation*

$$\frac{\partial \Phi_t(z)}{\partial t} = H(t, \Phi(t)), \quad \Phi_0(z) = z \in \mathbb{H}_n, \tag{4.10}$$

which gives a family $\{\Phi_t\}_{t\geq 0}$ of univalent self-mappings of $\mathbb{H}_n$, see Theorem 1.4 in [AB11]. A slight variation of the proof of Theorem 4.5 shows that $\Phi_t \in \mathfrak{P}_n$ for all $t \geq 0$ also in this case.

Question 4.7. *Let $f \in \mathfrak{P}_1$. In [GB92], Section 4, it is shown that there exists a $\mathcal{K}(\mathbb{H}, c)-$Herglotz vector field H and a time $T \geq 0$ such that $f = \Phi_T$, where $\{\Phi_t\}_{t\geq 0}$ is the solution of equation* (4.10). *What can be said in the higher dimensional case?*

4.3 On the behavior of iterates

Let $F : \mathbb{B}_n \to \mathbb{B}_n$ be holomorphic. We say that $p \in \overline{\mathbb{B}_n}$ is the Denjoy-Wolff point of F if $F^n \to p$ for $n \to \infty$ locally uniformly. The basic results about the behavior of the iterates F^n for $n \to \infty$ can be found in [Aba89], Chapter 2.2. In particular we have (Theorem 2.2.31):

$$F \text{ has a Denjoy-Wolff point on the boundary } \partial\mathbb{B}_n \iff F \text{ has no fixed points.} \tag{4.11}$$

Now let $f \in \mathfrak{P}_n$. For $n = 1$, f has the Denjoy-Wolff point ∞ if f is not the identity: As f is not an elliptic automorphism, the classical Denjoy-Wolff theorem implies that f has a Denjoy-Wolff point. This point has to be ∞, e.g. because of Proposition 4.3 c).
Next we will show that this is also true in higher dimensions, provided that f extends smoothly to the boundary point ∞. There are different possible definitions of smoothness of f near ∞. We will use the following one: Let $H(z) = f(z) - z$, and denote by $G : \mathbb{B}_n \to \mathbb{C}^n$ the corresponding generator on $\mathbb{B}_n$, i.e. we have $H(z) = (C^{-1})'(C(z)) \cdot G(C(z))$ and a small computation shows

$$H_1(z) = -\frac{i}{2}(z_1 + i)^2 \cdot G_1(C(z)).$$

Our smoothness condition will be that G_1 has a C^3-extension to e_1, i.e. we can write

$$G_1(z) = \sum_{\substack{k_1+\ldots+k_n\leq 3\\ k_1,\ldots,k_n\geq 0}} a_{k_1,\ldots,k_n}(z_1-1)^{k_1}\cdot z_2^{k_2}\cdot\ldots\cdot z_n^{k_n} + \mathcal{O}(\|z-e_1\|^3),$$

which translates to

$$H_1(z) = -\frac{i}{2}(z_1+i)^2\cdot \sum_{k_1+\ldots+k_n\leq 3} a_{k_1,\ldots,k_n}\left(\frac{-2i}{z_1+i}\right)^{k_1}\cdot\left(\frac{2z_2}{z_1+i}\right)^{k_2}\cdot\ldots\cdot\left(\frac{2z_n}{z_1+i}\right)^{k_n} + \mathcal{O}(\|C(z)-e_1\|^3),$$

or

$$\begin{aligned} H_1(z) &= b_{0,\ldots,0}\cdot(z_1+i)^2 + (z_1+i)\cdot\sum_{k_1+\ldots+k_n=1} b_{k_1,\ldots,k_n} z_2^{k_2}\cdot\ldots\cdot z_n^{k_n}\\ &+ \sum_{k_1+\ldots+k_n=2} b_{k_1,\ldots,k_n} z_2^{k_2}\cdot\ldots\cdot z_n^{k_n} + (z_1+i)^{-1}\cdot\sum_{k_1+\ldots+k_n=3} b_{k_1,\ldots,k_n} z_2^{k_2}\cdot\ldots\cdot z_n^{k_n}\\ &+ \mathcal{O}(|z_1+i|^{-1}\cdot\|(1,z_2,\ldots,z_n)\|^3) \end{aligned} \tag{4.12}$$

for some coefficients $b_{k_1,\ldots,k_n} \in \mathbb{C}$.

Theorem 4.8. *Let $f \in \mathfrak{P}_n$, $f \neq id$, and assume that* (4.12) *is satisfied. Then ∞ is the Denjoy-Wolff point of f.*

Proof. Write $f(z) = z + H(z)$, where $H \in \mathcal{K}(\mathbb{H}_n, c)$ and $H = (H_1, \tilde{H})$. Let $\gamma \in \mathbb{C}^{n-1}$. If we can show that the slice $h_\gamma(\zeta) = H_1(\varphi(\zeta)) - 2i\overline{\gamma}^T\tilde{H}(\varphi_\gamma(\zeta))$ has no zeros, then we are done:
This implies that H has no zeros because of (3.7) and (3.8). Hence, f has no fixed points and (4.11) implies that f has a Denjoy-Wolff point. This point has to be ∞ because of Proposition 4.3 c).
Similarly to the proof of Theorem 4.4, equation (4.4), we have

$$\|\tilde{H}(z)\| \leq \frac{c}{2|u_{\mathbb{H}_n}(z)|^{3/2}},$$

and thus

$$\|\tilde{H}(\varphi_\gamma(\zeta))\| \leq \frac{c}{2|u_{\mathbb{H}_n}(\varphi_\gamma(\zeta))|^{3/2}} = \frac{c}{2\mathrm{Im}(\zeta)^{3/2}}.$$

Consequently, $\lim_{y\to\infty} y|\overline{\gamma}^T\tilde{H}(\varphi_\gamma(iy))| = 0$. On the other hand, we know from Proposition 3.4 that $h_\gamma \in \mathcal{K}(\mathbb{H}, c)$ which implies (see Remark 1.5)

$$\limsup_{y\to\infty} y|h_\gamma(iy)| = \limsup_{y\to\infty} y|H_1(\varphi(iy)) - 2i\overline{\gamma}^T\tilde{H}(\varphi_\gamma(iy))| \leq c,$$

which gives us

$$\limsup_{y\to\infty} |iy\cdot H_1(\varphi_\gamma(iy))| \leq c. \tag{4.13}$$

Now we use the assumption of the smoothness of H_1 :
Because of (4.13), all coefficients $b_{k_1,...,k_n}$ from (4.12) with $k_1 + ... + k_n \leq 2$ have to be 0. Thus, $\lim_{y\to\infty} iy \cdot H_1(\varphi_\gamma(iy)) =: K(\gamma)$ exists and is a polynomial in $\gamma = (\gamma_2, ..., \gamma_n)$:

$$K(\gamma) = \sum_{k_1+...+k_n=3} b_{k_1,...,k_n} \gamma_2^{k_2} \cdot ... \cdot \gamma_n^{k_n}.$$

As $K(\gamma)$ is bounded, it has to be constant.
If $K(\gamma) \equiv 0$, then all slices of H are zero, hence $H = 0$ by Remark 3.5 and f is the identity, a contradiction.
Hence $K(\gamma)$ is a non-zero constant and $h_\gamma(\zeta)$ is not identically zero, which implies (e.g. by using the representation (1.5)) that $h_\gamma(\zeta)$ has no zeros.

□

Question 4.9. *Is ∞ the Denjoy-Wolff point for every $f \in \mathfrak{P}_n$?*

5 Appendix

Here we prove Lemma (3.2):

Let $a \in \mathbb{C}, p, v \in \mathbb{C}^{n-1}$ and $z = (z_1, \tilde{z}) \in \mathbb{H}_n$. Then the following formulas hold:

$$\left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\|_{\mathbb{H}_n,z} = \frac{|a|}{|u_{\mathbb{H}_n}(z)|}, \tag{5.1}$$

$$\left\| \begin{pmatrix} 2i\overline{p}^T v \\ v \end{pmatrix} \right\|_{\mathbb{H}_n,z} = 2\frac{\sqrt{\|v\|^2 |u_{\mathbb{H}_n}(z)| + |\overline{(p-\tilde{z})}^T v|^2}}{|u_{\mathbb{H}_n}(z)|}, \tag{5.2}$$

$$\left\| \begin{pmatrix} a - 2i\overline{\tilde{z}}^T v \\ 0 \end{pmatrix} + \begin{pmatrix} 2i\overline{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n,z} = \left\| \begin{pmatrix} a - 2i\overline{\tilde{z}}^T v \\ 0 \end{pmatrix} \right\|^2_{\mathbb{H}_n,z} + \left\| \begin{pmatrix} 2i\overline{\tilde{z}}^T v \\ v \end{pmatrix} \right\|^2_{\mathbb{H}_n,z}. \tag{5.3}$$

Proof. We write $\tilde{z} = (z_2, ..., z_n), v = (v_2, ..., v_n), p = (p_2, ..., p_n)$.

An explicit formula of the Kobayashi metric for the unit ball is given in [AFH$^+$04], Theorem 3.4.[4] It coincides with the Bergman metric and by using the Cayley map we get the following formula for the upper half-space:

$$\|w\|^2_{\mathbb{H}_n,z} = w^T \cdot (g_{j,k})_{j,k} \cdot \overline{w},$$

where $w \in \mathbb{C}^n$ and $(g_{j,k})_{j,k}$ is an $n \times n$-matrix with

$$g_{j,k} = -4\frac{\partial^2}{\partial z_j\, \partial \bar{z}_k} \log(\mathrm{Im}(z_1) - \sum_{l=2}^n |z_l|^2),$$

and we get for $j, k \geq 2$:

$$\begin{aligned} g_{1,1} &= \frac{1}{u_{\mathbb{H}_n}(z)^2}, \qquad g_{1,k} = \frac{2iz_k}{u_{\mathbb{H}_n}(z)^2}, \qquad g_{j,1} = \frac{-2i\overline{z_j}}{u_{\mathbb{H}_n}(z)^2}, \\ g_{j,j} &= 4\frac{\mathrm{Im}(z_1) - \sum_{l=2,l\neq j}^n |z_l|^2}{u_{\mathbb{H}_n}(z)^2}, \qquad g_{j,k} = \frac{4z_k\overline{z_j}}{u_{\mathbb{H}_n}(z)^2},\ k \neq j. \end{aligned}$$

The formulas (5.1) and (5.2) are now straightforward calculations. We obtain

$$\|(a,0)\|_{\mathbb{H}_n,z} = \sqrt{(a,0) \cdot (g_{j,k})_{j,k} \cdot \overline{(a,0)^T}} = \sqrt{a \cdot g_{1,1} \cdot \overline{a}} = \frac{|a|}{|u_{\mathbb{H}_n}(z)|},$$

and

[4] Note, however, that the Kobayashi metric in [AFH$^+$04] differs by a factor of 2 from the one we are using here.

$$
\begin{aligned}
& u_{\mathbb{H}_n}(z)^2 \cdot \|(2i\overline{p}^T v, v)\|^2_{\mathbb{H}_n,z} = u_{\mathbb{H}_n}(z)^2 \cdot (2i\overline{p}^T v, v^T) \cdot (g_{j,k})_{j,k} \cdot \overline{(2i\overline{p}^T v, v^T)^T} \\
= \; & u_{\mathbb{H}_n}(z)^2 \cdot (\sum_{j=2}^n g_{j,j}|v_j|^2 + g_{1,1}|2i\overline{p}^T v|^2 + \sum_{j=2}^n g_{j,1} v_j \overline{2i\overline{p}^T v} + \sum_{k=2}^n g_{1,k}\overline{v_j} 2i\overline{p}^T v + \sum_{j,k\geq 2, j\neq k} g_{j,k} v_j \overline{v_k}) \\
= \; & 4\sum_{j=2}^n (\mathrm{Im}(z_1) - \|\tilde{z}\|^2) \cdot |v_j|^2 + 4\sum_{j=2}^n |z_j|^2 \cdot |v_j|^2 \\
+ \; & 4\sum_{j,k\geq 2}^n p_j \overline{p_k} v_j \overline{v_k} - 4\sum_{j,k\geq 2}^n \overline{z_j} p_k v_j \overline{v_k} - 4\sum_{j,k\geq 2}^n z_j \overline{p_k v_j} v_k + 4\sum_{j,k\geq 2, j\neq k}^n \overline{z_j} z_k v_j \overline{v_k} \\
= \; & 4\|v\|^2 \cdot |u_{\mathbb{H}_n}(z)| + 4\sum_{j=2}^n z_j \overline{z_j} v_j \overline{z_j} \\
+ \; & 4\sum_{j,k\geq 2}^n (p_j \overline{p_k} v_j \overline{v_k} - \overline{z_j} p_k v_j \overline{v_k} - z_j \overline{p_k v_j} v_k) + 4\sum_{j,k\geq 2, j\neq k}^n \overline{z_j} z_k v_j \overline{v_k} \\
= \; & 4\|v\|^2 \cdot |u_{\mathbb{H}_n}(z)| + 4\sum_{j,k\geq 2}^n (p_j \overline{p_k} v_j \overline{v_k} - \overline{z_j} p_k v_j \overline{v_k} - z_j \overline{p_k v_j} v_k + \overline{z_j} z_k v_j \overline{v_k}) \\
= \; & 4\|v\|^2 \cdot |u_{\mathbb{H}_n}(z)| + 4|\overline{(p-\tilde{z})}^T v|^2.
\end{aligned}
$$

For formula (5.3) we just need to show that $(2i\overline{\tilde{z}}^T v, v^T) \cdot (g_{j,k})_{j,k} \cdot \overline{(a - 2i\overline{\tilde{z}}^T v, 0)}^T = 0$. Indeed, we have

$$
u_{\mathbb{H}_n}(z)^2 \cdot (g_{j,k})_{j,k} \cdot \overline{(a - 2i\overline{\tilde{z}}^T v, 0)}^T = (\overline{a} + 2i\tilde{z}^T\overline{v}, -2i\overline{z_2 a} + 4\overline{z_2}\tilde{z}^T\overline{v}, ..., -2i\overline{z_n a} + 4\overline{z_n}\tilde{z}^T\overline{v})^T
$$

and

$$
\begin{aligned}
& (2i\overline{\tilde{z}}^T v, v^T)(\overline{a} + 2i\tilde{z}^T\overline{v}, -2i\overline{z_2 a} + 4\overline{z_2}\tilde{z}^T\overline{v}, ..., -2i\overline{z_n a} + 4\overline{z_n}\tilde{z}^T\overline{v})^T \\
= \; & 2i\overline{a}\overline{\tilde{z}}^T v - 4|\tilde{z}^T\overline{v}|^2 - 2i\overline{a}\overline{\tilde{z}}^T v + 4|\tilde{z}^T\overline{v}|^2 = 0.
\end{aligned}
$$

□

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